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Electrostatics of two suspended spheres

(Eletrostática de duas esferas suspensas)

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Although the working principle of a traditional electroscope with thin metal deflection foils is simple, one needs numerical methods to calculate its foil's deflection. If the electroscope is made of hanging spheres instead of foils, then it is possible to obtain an analytical solution. Since the separation of the charged spheres is of the order of their radius the spheres cannot be described as point charges. We apply the method of image charges to find the electrostatic force between the spheres and then we relate the voltage applied to their separation. We also discuss the similarity with the sphere-plane electrostatic problem. This approach can be used as an analytical solution for practical problems in the field of electrodynamics and its complexity is compatible with undergraduate courses.
Keywords: electroscope, electrometer, method of image, image charge, electrostatic, sphere-sphere.

Apesar da simplicidade do princípio de funcionamento do eletroscópio tradicional feito de folhas metálicas finas, é necessário métodos numéricos para calcular a deflexão das folhas. Se o eletroscópio for feito de esferas penduradas ao invés de folhas, então é possível obter uma solução analítica. Como a deflexão das esferas carregadas é da ordem dos seus raios, as esferas não podem ser descritas como cargas pontuais. Nós aplicamos o método das cargas imagens para encontrar a força eletrostática entre as esferas, e então, correlacionamos a voltagem aplicada com a separação. Discutimos também as similaridades com o problema eletrostático de uma esfera e um plano. Esta abordagem pode ser usada como uma solução analítica para problemas práticos em eletrodinâmica e sua complexidade e compatível com cursos de graduação.
Palavras-chave: eletroscópio, eletrômetro, método das imagens, carga imagem, eletrostática, esfera-esfera.

1. Introduction

The electroscope is an instrument presented to students in their introduction to electrostatics as a demonstration of the existence of electric charges (Fig. 1). Historically, this instrument was also important in the development of electricity [1]. If the electroscope is calibrated to provide the value of the charge (or the voltage) it is often called an *electrometer*. Usually, electroscopes are made of very light deflectable metal foils, in which it is impossible to calculate the charge they can store analytically. We describe here the properties of two suspended and electrically connected spheres that can move in a vertical plane only: in this case an analytical solution of the separation of the spheres can be found. We shall use the *method of images* to correlate the applied voltage to the deflection of the spheres. Applying this method is easier than solving Laplace's equation directly [2].

In the introduction courses to electrostatics, many exercises involve charged spheres. Usually the problem is simplified by regarding the spheres as point charges. We shall show that the error in the determination of the force between the spheres or any other physical quantity (applied voltage; total charge, etc) can be large, if the image charges are ignored. Moreover, in real applications the point charge approximation for spheres can usually not be applied. For example, in those funny hair bristle *Van de Graaff* generators for educational purposes (Fig. 2), the main sphere and the spherical accessories used in the apparatus are manipulated at small distances, so the effect of image charges is quite large. In these cases to evaluate the total charge, the voltage, or the force, etc, it is important to take these image charges into account.

2. Method of image charge

Figure 3 illustrates the method of image charge for a point charge close to a grounded sphere [3,4]. Consider the sphere with radius a centered at x_0 and the point charge, q_0, placed at -x_0. The solution for the electric

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potential can be determined as if an image charge q_1 was at position x_1. The point charge q_1 is virtual; the real charge induced is actually distributed at the surface of the sphere. The solid and dashed lines represent the real and virtual field lines respectively. Click the URL in Ref. [5] to watch an animation of the electric field lines as a function of x_0. The potential ϕ must be identically zero everywhere at the surface of the sphere particularly

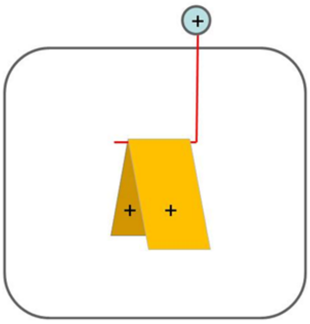

Figure 1 - The Electroscope of leaves; the deflection of the foils is impossible to be calculated analytically.

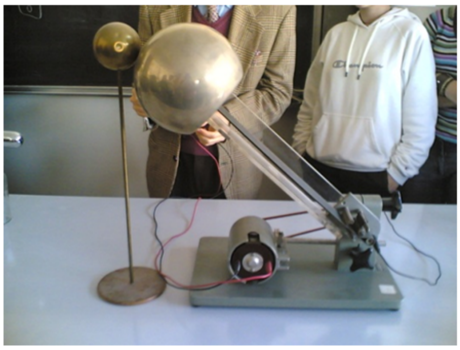

Figure 2 - Van de Graaff generator showing the main sphere and an accessory sphere at small distance compared to their diameters. This is usually the case in real electrostatic systems and the image charges must be taken into account.

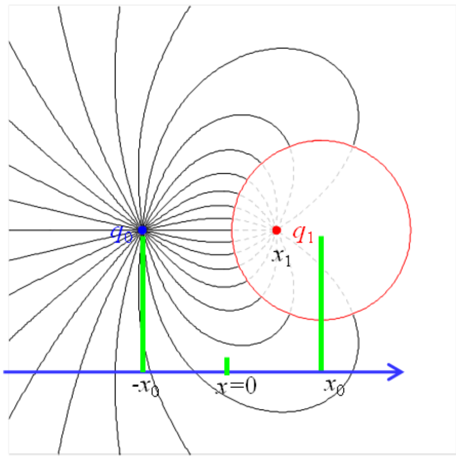

Figure 3 - Illustration of the method of images: a real charge q_0 induces an opposite charge q_1 in a grounded sphere. The charge induced is as if q_1 be a point charge located at x_1.

$$\phi(x_0 - a) = k\left(\frac{q_0}{2x_0 - a} + \frac{q_1}{x_1 - (x_0 - a)}\right) = 0, \quad (1)$$

$$\phi(x_0 + a) = k\left(\frac{q_0}{2x_0 + a} + \frac{q_1}{x_0 + a - x_1}\right) = 0, \quad (2)$$

where k is dielectric constant of vacuum. Also $\phi(\infty) = 0$ by definition. Solving the system above we have

$$q_1 = -\frac{a}{2x_0} q_0, \quad (3)$$

$$x_1 = x_0 - \frac{a^2}{2x_0}. \quad (4)$$

3. Electroscope with two spheres

Assume an electroscope made of two identical conductive spheres. The spheres have radius a, mass m and they hang from two massless conductive wires; so, the spheres are electrically connected. The wires have length L and separation $2a$. The capacitance of the wires is negligible so only the spheres store charge. When the electroscope is neutral (Fig. 4a) the spheres touch each other, but with no contact force. In this situation, we assume the voltage $\phi = 0$ at the spheres. If a voltage $\phi = V$ is applied (Fig. 4b), the spheres becomes separated by d. In this problem d is the measurable variable so the problem is to find V as a function of the separation d.

In calculating the electric potential one must define its value at some point in the space in the physical system. Usually it is convenient to make $\phi = 0$ at infinity as is the case here, so when the spheres are uncharged ϕ must equal zero also at the spheres. Then, there is no electric field because there is no potential difference in the system.

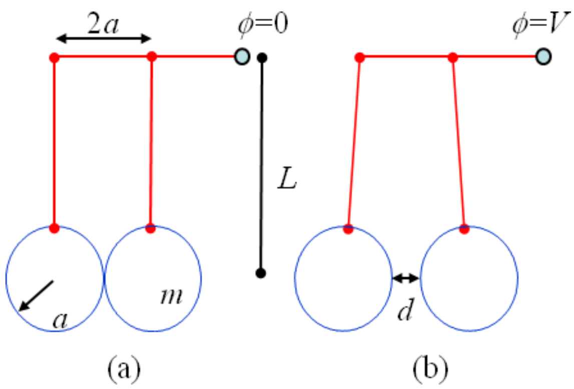

Figure 4 - Schematic diagram of the electroscope (a) uncharged and (b) charged. This electroscope with spheres makes possible to evaluate the applied voltage and charge.

4. Solution for the electric force

A simplification of the electroscope is depicted in Fig. 5, in which only the spheres are shown. The sphere-sphere problem is very similar to the sphere-plane problem that we described in detail in Ref. [6]. Any difference between these two solutions will be discussed.

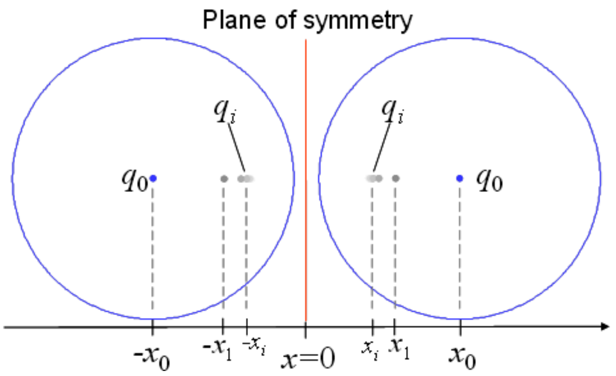

Figure 5 - When two spheres of charge q_0 are approximated each one generates a series of image charges $\{q_i\}$ into the other.

Starting from central charges q_0, each of these induces q_1 at the other sphere, which in turn induces back a charge q_2 and so on. The distribution of the total charge, q, is equivalent to a series of point charges q_i. The solution is symmetric with respect to the plane $x = 0$.

Applying the procedure of section 2 the positions and the magnitude of the image charges are given by the recurrent relations

$$x_{i>0} = x_0 - \frac{a^2}{x_0 + x_{i-1}} \tag{5}$$

and

$$q_i = \frac{-a}{x_0 + x_{i-1}} q_{i-1}, \tag{6}$$

where

$$x_0 = d/2 + a \tag{7}$$

and

$$q_0 = \frac{aV}{k}. \tag{8}$$

It is convenient to define a dimensionless relative charge, ξ_i, as the ratio between the i^{th} image charge and the central charge q_0

$$\xi_{i>0} = \frac{q_i}{q_0} = \frac{-a}{x_0 + x_{i-1}} \xi_{i-1}, \tag{9}$$

with $\xi_0 = 1$. The negative sign in Eqs. (6) and (9) does not appear in the sphere-plane problem, because there the electrodes have opposite charges. Here, the image charges in the spheres have alternating signs, while the image charges in the sphere-plane problem are all positive in the sphere and all negative in the plane. The total charge in each sphere is given by

$$q = \sum_{i=0}^{\infty} q_i = q_0 \sum_{i=0}^{\infty} \xi_i. \tag{10}$$

Let the parameter ξ, with no index be the total charge compared to the central charge as

$$\xi = \sum_{i=0}^{\infty} \xi_i. \tag{11}$$

Now that x_i and ξ_i are determined, the derivation of the force, F, between the spheres is straightforward. The force between a point charge with index i in one sphere and a point charge with index j in the other sphere is given by Coulomb's law

$$F_{ij} = k q_0^2 \frac{\xi_i \xi_j}{(x_i + x_j)^2}. \tag{12}$$

Replacing q_0 for the expression in Eq. (8) the total force is the summation over all pairs (i,j)

$$F = \frac{(aV)^2}{k} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{\xi_i \xi_j}{(x_i + x_j)^2}. \tag{13}$$

Figure 6 shows the deflection of one sphere's center when the electroscope is charged. The equilibrium among the electrical force, the weight (P) and the tension in the wire correlates V and d according to

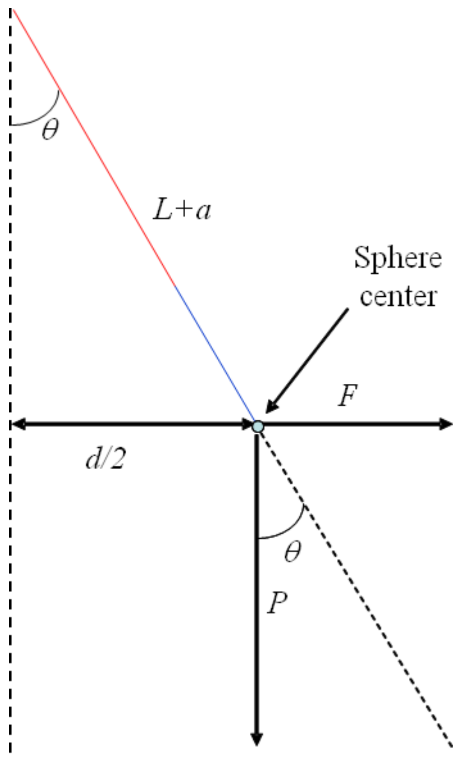

Figure 6 - Equilibrium of forces in the charged electroscope.

$$\tan(\theta) = \frac{F}{P} = \frac{d/2}{\sqrt{(L+a)^2 - (d/2)^2}} \Rightarrow$$
$$F = mg\frac{d/2}{\sqrt{(L+a)^2 - (d/2)^2}}, \quad (14)$$

where g is the acceleration of gravity. By combining Eqs. (13) and (14) we get an expression for V

$$V = \sqrt{\frac{kmg}{a^2 \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{\xi_i \xi_j}{(x_i+x_j)^2}} \frac{d/2}{\sqrt{(L+a)^2 - (d/2)^2}}}. \quad (15)$$

5. Convergence analysis

Table 1 shows the values for x_i and ξ_i for $a = 1$ arbitrary length unit (a.u). and $d = 0.1$ a.u. The image charges positions tend fast to $x_\infty = \sqrt{x_0^2 - a^2}$ and the relative charges magnitudes tend to $\xi_\infty = 0$. The greater is d, the faster is the convergence [6].

Table 1 - Relative charge magnitude and their positions for $d/a = 0.1$.

$\xi_0 = 1$	$x_0 = 1.05$
$\xi_1 = -0.47619$	$x_1 = 0.57381$
$\xi_2 = 0.29326$	$x_2 = 0.43416$
$\xi_3 = -0.19759$	$x_3 = 0.37622$
$\xi_4 = 0.13854$	$x_4 = 0.34885$
$\xi_5 = -0.09904$	$x_5 = 0.33513$
$\xi_6 = 0.07150$	$x_6 = 0.32804$
$\xi_\infty = 0$	$x_\infty = 0.320156$

Figure 7 shows V for several values of d in a.u., $a = 1$ a.u., L=10 a.u. and $m = 10^{-4}$ kg. The right axis of Fig. 7 shows the normalized total charge ξ. Even for d as large as 5 times the radius of the spheres, ξ is yet 87%, *i.e.* the total charge is only 87% of the central charge. This means that the complementary 13% is typically the error one would have in F by ignoring the image charges in the spheres. It can be seen in Fig. 7 that the voltages to deflect the spheres are quite high. However, rubbed objects can easily obtain voltages greater than 100 kV. Such high voltages may cause a sudden discharge of the electroscope, since the dielectric break down limit in air is $E_{max} \cong 3$ MV/m; so, arcing will occur if an external object is brought close to the spheres, say at a distance < 1 mm [7]. An upper limit of the measurable voltage can be estimated from the field of one of the spheres: $E_{max} = V_{max}/a \Rightarrow V_{max} = 30$ kV. This value is overestimated by ~10% according to the error discussed above. Anyway, it gives a good estimative for the maximum measurable voltage. For $a = 1$ cm, one gets maximum measurable separation of $d_{max} \cong$ 4.5 cm. Figure 8 illustrates the norm of the electric field nearby the spheres for $d = 0.5a$. This figure was made using the expressions for the electric field presented in the appendix. The electric field is maximum at the outer poles of the spheres in the blue regions.

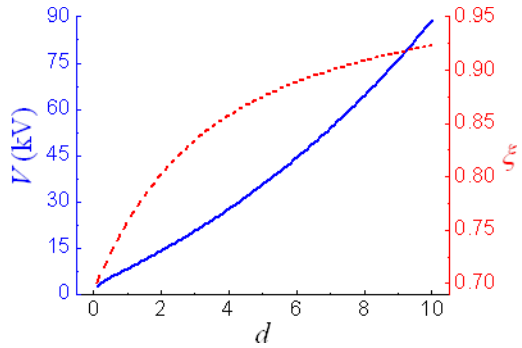

Figure 7 - Curves for the voltage V (solid line) and for the relative charge ξ (dashed curve).

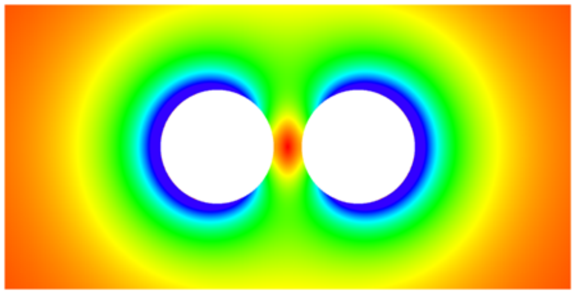

Figure 8 - Density plot of the electric field distribution for $d = 0.5a$. The shades of blue indicate strong electric field regions.

6. Conclusion

We obtained an analytical solution for an electroscope of two suspended spheres using the method of image charges. This is a simple and representative solution: it can easily be generalized to more spheres, spheres of different sizes and dielectric spheres. Not taking the induced image charges into account in this problem will lead to a significant error in the evaluation of the total charge and voltage. Our calculation also indicates that an electroscope with two metal spheres needs rather high voltages to get deflection; however, those high voltages needed are easily obtained by rubbing objects. The maximum deflection is obtained when the electric field strength close to the spheres reaches the dielectric breakdown limit in air.

Appendix: formulary

Here we list the equations used in this article and also equations related to other physical quantities such as the electric field, potential distributions and capacitance of the system that might interest the reader.

Image charge positions	$x_{i>0} = x_0 - \frac{a^2}{x_0+x_{i-1}}$
Initial (central) charge	$q_0 = \frac{aV}{k}$
Image charge magnitude	$q_i = \frac{-a}{x_0+x_{i-1}} q_{i-1}$
Relative charge magnitude q_i/q_0	$\xi_{i>0} = \frac{q_i}{q_0} = \frac{-a}{x_0+x_{i-1}} \xi_{i-1}$
Total charge	$q = q_0 \sum\limits_{i=0}^{\infty} \xi_i$
Potential (valid outside the spheres)	$\phi(r,x) = aV \sum\limits_{i=0}^{\infty} \frac{\xi_i}{\left[(x-x_i)^2+r^2\right]^{1/2}} + \frac{\xi_i}{\left[(x+x_i)^2+r^2\right]^{1/2}}$
Electric field: r component (valid outside the spheres)	$E_r(r,x) = aVr \sum\limits_{i=0}^{\infty} \frac{\xi_i}{\left[(x-x_i)^2+r^2\right]^{3/2}} + \frac{\xi_i}{\left[(x+x_i)^2+r^2\right]^{3/2}}$
Electric field: x component (valid outside the spheres)	$E_x(r,x) = aV \sum\limits_{i=0}^{\infty} \frac{\xi_i(x-x_i)}{\left[(x-x_i)^2+r^2\right]^{3/2}} + \frac{\xi_i(x+x_i)}{\left[(x+x_i)^2+r^2\right]^{3/2}}$
Capacitance	$C = \frac{2a}{k} \sum\limits_{i=0}^{\infty} \xi_i$
Force	$F = \frac{(aV)^2}{k} \sum\limits_{i=0}^{\infty} \sum\limits_{j=0}^{\infty} \frac{\xi_i \xi_j}{(x_i+x_j)^2}$

7. Acknowledgments

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